Absolute purity in motivic homotopy theory

Fangzhou Jin joint work with F. Deglise, J. Fasel and A. Khan

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The absolute purity conjecture

Grothendieck's **absolute (cohomological) purity conjecture** (SGA5, Expose I 3.1.4) is the following statement: if $i : Z \to X$ is a closed immersion between noetherian regular schemes of pure codimension $c, n \in \mathcal{O}(X)$ and $= \mathbb{Z} = n\mathbb{Z}$, then the etale cohomology sheaf supported in Z with values in can be computed as

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A short history of the proof

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 Based on Thomason's method + rigidity for algebraic
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- Show that the six functors on the derived category of etale sheaves preserve constructible objects.
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- Our work: study absolute purity in the framework of motivic homotopy theory.
- Main result: the absolute purity in motivic homotopy theory is satis ed with rational coe cients in mixed characteristic.

Grothendieck's absolute purity conjecture Motivic homotopy theory The fundamental class Grothendieck's absolute purity conjecture Motivic homotopy theory The fundamental class Grothendieck's absolute purity conjecture Motivic homotopy theory The fundamental class Absolute purity in m 1 - . 4 TJ32 - . 4 4Tdnomor1

Motivic homotopy theory

- The motivic homotopy theory or A¹-homotopy theory is introduced by Morel and Voevodsky (1998) as a framework to study cohomology theories in algebraic geometry, by importing tools from algebraic topology
- Idea: use the a ne line A¹ as a substitute of the unit interval to get an algebraic version of the homotopy theory
- Can be used to study cohomology theories such as algebraic *K*-theory, Chow groups (motivic cohomology) and many others
- Advantage: has many a lot of structures coming from both topological and algebraic geometrical sides

Aspects of applications in various domains

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- Non-commutative geometry and singularity categories (Tabuada, Blanc-Robalo-Toen-Vezzosi)

Some topological background

• A **spectrum** \mathbb{E} is a sequence $(E_n)_{n \ge \mathbb{N}}$ of pointed spaces (e.g. CW-complexes or simplicial sets) together with continuous maps $_n : S^1 \land E_n \to E_{n+1}$ called **suspension maps**

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- Examples: Suspension spectra ${}^7 X$ for $X \in Top$, in particular sphere spectrum S; HA Eilenberg-Mac Lane spectrum for a ring A; MU complex cobordism spectrum
- From an ∞-categorical point of view, the category of spectra is the stabilization of the category of spaces, and is the universal stable (triangulated) category

The unstable motivic homotopy category

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- Bigraded \mathbb{A}^1 -homotopy sheaves: for $X \in \mathbf{H}(S)$, $\mathbb{A}^1_{a,b}(X)$ is the Nisnevich sheaf on Sm_S associated to the presheaf

$$U \mapsto [U \wedge S^{a \ b} \wedge \mathbb{G}_{m}^{b}; X]_{\mathbf{H}_{\bullet}(S)}$$

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- SH(S) is the universal stable ∞-category which satis es Nisnevich descent and A¹-invariance (Robalo, Drew-Gallauer)

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- Milnor-Witt spectrum H_{MW}ℤ represents Milnor-Witt motivic cohomology/higher Chow-Witt groups (Deglise-Fasel)

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- The 1-line is also computed (Rondigs-Spitzweck- stvaer):

$$0 \rightarrow K_{2 n}^{M} = 24 \rightarrow n+1, n(\mathbb{1}_{k}) \rightarrow n+1, nf_{0}(\mathbf{KQ})$$

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• They satisfy formal properties axiomatizing important theorems such as duality, base change and localization.

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- This construction passes through \mathbb{P}^1 -stabilization and de nes a \otimes -invertible object in $\mathbf{SH}(X)$, and the map $V \mapsto Th(V)$ extends to a map $\mathcal{K}_0(X) \to \mathbf{SH}(X)$
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- Relative purity (Ayoub): $f : X \to Y$ smooth morphism with tangent bundle T_f , then $f^! \simeq Th(T_f) \otimes f$
- In the presence of an *orientation*, we recover the usual relative purity

Orientations

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- Examples: **H**Z, **KGL**, **MGL**, or the spectrum representing etale cohomology
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- A theory of *fundamental classes* aims at establishing a cohomological intersection theory
- For oriented spectra, Deglise de ned fundamental classes using Chern classes

Bivariant groups

For f : X → S be a separated morphism of nite type,
 v ∈ K₀(X) and E ∈ SH(S), de ne the E-bivariant groups (or Borel-Moore E-homology) as

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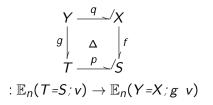
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• If S is a eld and $\mathbb{E} = \mathbf{H}\mathbb{Z}$, then $\mathbb{E}_i(X=S;v) = CH_r(X;i)$ are the higher Chow groups, where r is the virtual rank of v

Grothendieck's absolute purity conjecture Motivic homotopy theory The fundamental class

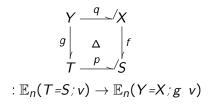
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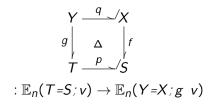


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• Product: if \mathbb{E} has a ring structure, $X \xrightarrow{f} Y \xrightarrow{g} S$

$$\mathbb{E}_m(X=Y;w)\otimes\mathbb{E}_n(Y=S;v)\to\mathbb{E}_{m+n}(X=S;w+f~v)$$

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- Morally, these operations contain the information of \intersecting cycles over X with Y"
- The construction uses the deformation to the normal cone

Euler class and excess intersection formula

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• Motivic Gauss-Bonnet formula (Levine, Deglise-J.-Khan) For $p: X \rightarrow S$ a smooth and proper morphism

$$(X=S) = p \ e(T_p)$$

where (X=S) is the categorical Euler characteristic

The absolute purity property

• We say that an absolute spectrum \mathbb{E} satis es **absolute purity** if for any closed immersion $i : Z \to X$ between regular schemes, the purity transformation $\mathbb{E}_Z \otimes \text{Th}(_f) \to f^! \mathbb{E}_X$ is an isomorphism

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 From this property Cisinski-Deglise deduce that the rational motivic Eilenberg-Mac Lane spectrum HQ also satis es absolute purity, mainly because HQ is a direct summand of KGLQ by the Grothendieck-Riemann-Roch theorem

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- \bullet The +-part $\mathbb{1}_{+,\mathbb{Q}}$ agrees with $\textbf{H}\mathbb{Q}$ (Cisinski-Deglise)

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First reductions:

- The \switching factors" endomorphism of P¹ ∧ P¹ induces a decomposition of the sphere spectrum 1_Q into the direct sum of the plus-part 1_{+,Q} and the minus-part 1_{,Q} (Morel)
- The +-part $\mathbb{1}_{+,\mathbb{Q}}$ agrees with $\textbf{H}\mathbb{Q}$ (Cisinski-Deglise)
- Therefore it su ces to show that the minus part satis es aboslute purity

The rst proof

 By a devissage theorem of Schlichting and an argument similar to the case of KGL, one can show that the Hermitian K-theory spectrum KQ satis es aboslute purity

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- Similar to the Chern character, the Borel character (de ned by Deglise-Fasel) induces a decomposition of KQ_Q, where 1 ,_Q can be identi ed as a direct summand
- This proves the absolute purity of $\mathbb{1}_{\mathbb{Q}}$ when 2 is invertible on the base scheme, since KQ is only well-de ned in this case

The second proof

• For every scheme X, denote by $_X : X_{\mathbb{Q}} = X \times_{\mathbb{Z}} \mathbb{Q} \to X$

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- The key lemma then reduces the absolute purity of 1 ,Q in mixed characteristic to the case of Q-schemes, which can be proved using Popescu's theorem: a closed immersion of a ne regular schemes over a perfect eld is a limit of closed immersions of smooth schemes

- Our method can be used to deduce the following new results in mixed characteristic:
 - The six functors preserve constructible objects in the rational stable motivic homotopy category $\textbf{SH}(\cdot;\mathbb{Q})$
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- Related work: absolute purity of the sphere spectrum over a Dedekind domain (Frankland-Nguyen-Spitzweck, work in progress)

Thank you!

Fangzhou Jin joint work with F. Déglise, J. Fasel and A. Khan Absolute purity in motivic homotopy theory